

SMEARING OF OBSERVABLES AND SPECTRAL MEASURES ON QUANTUM STRUCTURES

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ABSTRACT. An observable on a quantum structure is any σ -homomorphism of quantum structures from the Borel σ -algebra of the real line into the quantum structure which is in our case a monotone σ -complete effect algebras with the Riesz Decomposition Property. We show that every observable is a smearing of a sharp observable which takes values from a Boolean σ -subalgebra of the effect algebra, and we prove that for every element of the effect algebra there is its spectral measure.

1. INTRODUCTION

D-posets introduced by Kôpka and Chovanec [KoCh] and effect algebras introduced by Foulis and Bennet [FoBe] became the last two decades very important quantum structures which model quantum mechanical events. Both structures are partial algebraic structures. Subtraction of two comparable events is a basic notion for D-posets, and addition of two mutually excluding events is a basic one for effect algebras. We recall that both structures are equivalent as it was mentioned in [FoBe]. In our paper we will deal only with effect algebras.

We note that a prototypical example of effect algebras, important mainly for measurements in Hilbert space quantum mechanics, is the system $\mathcal{E}(H)$ of all Hermitian operators of a (real, complex or quaternionic) Hilbert space H that lie between the zero and the identity operator. The system $\mathcal{E}(H)$ is used for modeling unsharp observables via POV-measures (= positive operator valued measure) in measurements in quantum mechanics.

To describe a measurement on a quantum structure, we use the notion of an observable. This is an analogue of a random variable in a classical measurement. In our case it is simply a σ -homomorphism of effect algebras from the Borel σ -algebra $\mathcal{B}(\mathbb{R})$ of the real line \mathbb{R} into the quantum structure.

¹Keywords: Effect algebra, observable, smearing of observables, monotone σ -completeness, state, Loomis-Sikorski theorem, effect-tribe, Riesz decomposition property, spectral measure

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Observables as an important tool of quantum structures are intensively studied by many authors. A functional calculus of observables on D-posets is presented in [KoCh]. The series of papers [Pul, JPV, JPV1] is dedicated to observables studied on lattice effect algebras and σ -MV-algebras exhibiting spectral properties and smearing of fuzzy observables by sharp observables using a kind of a Markov kernel. In [DvKu], it was shown that in many important structures, a partial information on an observable known only on intervals of the form $(-\infty, t)$, $t \in \mathbb{R}$, is sufficient to derive the whole information on the observable.

The main tool in our research will be applications of the Loomis-Sikorski Theorem for monotone σ -complete effect algebras with the Riesz Decomposition Property (RDP) proved in [BCD]. This Theorem says that our structure is a σ -homomorphic image of a monotone σ -complete effect algebra of fuzzy sets where all algebraic operations are defined by points. This generalizes analogous results proved for a special case of monotone σ -complete effect algebras, called σ -complete MV-algebras, see [Mun, Dvu1]. We recall that RDP is a special type of distributivity which in our case means the possibility of performing a joint refinement of two decompositions. It has an important consequence that an effect algebra with RDP is always an interval in a partially ordered group with strong unit (= order unit), see [Rav].

The present paper is inspired by the research in [Pul]. We have two aims. First, we show that every observable of a monotone σ -complete effect algebra M with RDP is a smearing of a sharp observable, where a sharp observable means that its values are in the biggest Boolean σ -subalgebra of M . Second, we show that every element a of M admits a spectral measure Λ_a , which is a sharp observable concentrated on the real interval $[0, 1]$. In the language of spectral theory of self-adjoint operators, Λ_a is the spectral measure of the element a . Analogous questions were inspected also in [Pul] for σ -complete MV-algebras and unital Dedekind σ -complete ℓ -groups [Pul1].

The paper is organized as follows. Section 2 is gathering necessary notions on effect algebras. Section 3 is studying a canonical representation as well as regular representations of monotone σ -complete effect algebras with RDP. Finally, Section 4 presents the main results on smearing of observables by sharp ones, and spectral measures of elements are established.

2. BASIC NOTIONS OF EFFECT ALGEBRAS

We recall that according to [FoBe], an *effect algebra* is a partial algebra $M = (M; +, 0, 1)$ with a partially defined operation $+$ and two constant elements 0 and 1 such that, for all $a, b, c \in M$,

- (i) $a + b$ is defined in M if and only if $b + a$ is defined, and in such a case $a + b = b + a$;
- (ii) $a + b$ and $(a + b) + c$ are defined if and only if $b + c$ and $a + (b + c)$ are defined, and in such a case $(a + b) + c = a + (b + c)$;
- (iii) for any $a \in M$, there exists a unique element $a' \in M$ such that $a + a' = 1$;
- (iv) if $a + 1$ is defined in M , then $a = 0$.

If we define $a \leq b$ if and only if there exists an element $c \in M$ such that $a + c = b$, then \leq is a partial ordering on M , and we write $c := b - a$; then $a' = 1 - a$ for any $a \in M$. As a basic source of information about effect algebras we can recommend the monograph [DvPu]. An effect algebra is not necessarily a lattice. We recall

that a *homomorphism* is any mapping of two effect algebras which preserves 1 and the addition +.

We show two kinds of important effect algebras. (1) If M is a system of fuzzy sets on Ω , that is $M \subseteq [0, 1]^\Omega$, such that (i) $1 \in M$, (ii) $f \in M$ implies $1 - f \in M$, and (iii) if $f, g \in M$ and $f(\omega) \leq 1 - g(\omega)$ for any $\omega \in \Omega$, then $f + g \in M$, then M is an effect algebra of fuzzy sets which is not necessarily a Boolean algebra as well as not a lattice. (2) If G is a partially ordered group written additively, $u \in G^+$, then $\Gamma(G, u) := [0, u] = \{g \in G : 0 \leq g \leq u\}$ is an effect algebra with $0 = 0$, $1 = u$ and $+$ is the group addition of elements if it exists in $\Gamma(G, u)$.

We say that an effect algebra M satisfies the Riesz Decomposition Property (RDP for short) if for all $a_1, a_2, b_1, b_2 \in M$ such that $a_1 + a_2 = b_1 + b_2$, there are four elements $c_{11}, c_{12}, c_{21}, c_{22}$ such that $a_1 = c_{11} + c_{12}$, $a_2 = c_{21} + c_{22}$, $b_1 = c_{11} + c_{21}$ and $b_2 = c_{12} + c_{22}$.

We note that an element of an effect algebra M is said to be *sharp* if $a \wedge a'$ exists in M and $a \wedge a' = 0$. Let $\text{Sh}(M)$ be the set of sharp elements of M . Then (i) $0, 1 \in \text{Sh}(M)$, (ii) if $a \in \text{Sh}(M)$, then $a' \in \text{Sh}(M)$. If M is a lattice effect algebra, then $\text{Sh}(M)$ is an orthomodular lattice which is a subalgebra and a sublattice of M , [JeRi]. If an effect algebra M satisfies RDP, then by [Dvu2, Thm 3.2], $\text{Sh}(M)$ is even a Boolean algebra, and an element a is sharp iff $a \wedge a'$ is defined in M and $a \wedge a' = 0$.

An effect algebra M is *monotone σ -complete* if, for any sequence $a_1 \leq a_2 \leq \dots$, the element $a = \bigvee_n a_n$ is defined in M (we write $\{a_n\} \nearrow a$). We recall that a mapping $x : \mathcal{B}(\mathbb{R}) \rightarrow M$ is said to be an *observable* on M if (i) $x(\mathbb{R}) = 1$, (ii) if E and F are mutually disjoint Borel sets, then $x(E \cup F) = x(E) + x(F)$, where $+$ is the partial addition on M , and (iii) if $\{E_i\}$ is a sequence of Borel sets such that $E_i \subseteq E_{i+1}$ for every i and $E = \bigcup_i E_i$, then $x(E) = \bigvee_i x(E_i)$. In other words, an observable is a σ -homomorphism of effect algebras.

An *effect-tribe* is any system \mathcal{T} of fuzzy sets on $\Omega \neq \emptyset$ such that (i) $1 \in \mathcal{T}$, (ii) if $f \in \mathcal{T}$, then $1 - f \in \mathcal{T}$, (iii) if $f, g \in \mathcal{T}$, $f \leq 1 - g$, then $f + g \in \mathcal{T}$, and (iv) for any sequence $\{f_n\}$ of elements of \mathcal{T} such that $f_n \nearrow f$ (pointwise), then $f \in \mathcal{T}$. It is evident that any effect-tribe is a monotone σ -complete effect algebra. We recall that e.g. $\mathcal{E}(H)$ can be represented as an effect-tribe, but RDP fails for it.

A very important subclass of effect algebras is the class of MV-algebras introduced by Chang [Cha].

We recall that an MV-algebra is an algebra $M = (M; \oplus, *, 0, 1)$ of type (2,1,0,0) such that, for all $a, b, c \in M$, we have

- (i) $a \oplus b = b \oplus a$;
- (ii) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$;
- (iii) $a \oplus 0 = a$;
- (iv) $a \oplus 1 = 1$;
- (v) $(a^*)^* = a$;
- (vi) $a \oplus a^* = 1$;
- (vii) $0^* = 1$;
- (viii) $(a^* \oplus b)^* \oplus b = (a \oplus b^*)^* \oplus a$.

If we define a partial operation $+$ on M in such a way that $a + b$ is defined in M if and only if $a \leq b^*$ and we set $a + b := a \oplus b$, then $(M; +, 0, 1)$ is an effect algebra with RDP which is a distributive lattice.

We recall that a *tribe* on $\Omega \neq \emptyset$ is a collection \mathcal{T} of fuzzy sets from $[0, 1]^\Omega$ such that (i) $1 \in \mathcal{T}$, (ii) if $f \in \mathcal{T}$, then $1 - f \in \mathcal{T}$, and (iii) if $\{f_n\}$ is a sequence from \mathcal{T} , then $\min\{\sum_{n=1}^\infty f_n, 1\} \in \mathcal{T}$. A tribe is always a σ -complete MV-algebra of fuzzy sets where MV-operations are defined by points.

3. LOOMIS–SIKORSKI THEOREM

In this section, we study representations of σ -complete effect algebra with RDP: a canonical representation and regular ones. A basic tool of investigation in our study is an application of the Loomis-Sikorski Theorem of monotone σ -complete effect algebras with RDP proved in [BCD]:

Theorem 3.1. *Every monotone σ -complete effect algebra with RDP is a σ -epimorphic image of an effect-tribe with RDP.*

For σ -complete MV-algebras, we have a Loomis-Sikorski type representation which was proved independently in [Mun, Dvu1]:

Theorem 3.2. *Every σ -complete MV-algebra is a σ -epimorphic image of a tribe.*

The proofs of these results proved in [BCD, Dvu1] used the notion of states, analogues of probability measures.

We recall that a *state* on an effect algebra M is any mapping $s : M \rightarrow [0, 1]$ such that (i) $s(1) = 1$ and (ii) $s(a + b) = s(a) + s(b)$ whenever $a + b$ is defined in M . We denote by $\mathcal{S}(M)$ the set of all states on M . It can happen that $\mathcal{S}(M)$ is empty, see e.g. [DvPu, Ex 4.2.4]. But if M satisfies RDP, $\mathcal{S}(M)$ is nonempty, see [Rav] and [Goo, Cor. 4.4]. In particular, every MV-algebra admits a state. We recall that $\mathcal{S}(M)$ is always a convex set. A state s is said to be *extremal* if $s = \lambda s_1 + (1 - \lambda)s_2$ for $\lambda \in (0, 1)$ implies $s = s_1 = s_2$. We denote by $\partial_e \mathcal{S}(M)$ the set of all extremal states of $\mathcal{S}(M)$. We say that a net of states, $\{s_\alpha\}$, on M *weakly converges* to a state s on M if $s_\alpha(a) \rightarrow s(a)$ for any $a \in M$. In this topology, $\mathcal{S}(M)$ is a compact Hausdorff topological space and every state on M lies in the weak closure of the convex hull of the extremal states as it follows from the Krein-Mil'man theorem. Hence, $\mathcal{S}(M)$ is empty iff so is $\partial_e \mathcal{S}(M)$.

If $\mathcal{S}(M)$ is non-void, given an element $a \in M$, we define a function $\hat{a} : \mathcal{S}(M) \rightarrow [0, 1]$ by

$$\hat{a}(s) := s(a), \quad s \in \mathcal{S}(M).$$

Then \hat{a} is a continuous affine function on $\mathcal{S}(M)$.

It is important to note that if M is an MV-algebra, $\partial_e \mathcal{S}(M)$ is always a compact set. In general, this is not true for every effect algebra. However, a delicate result of Choquet [Alf, page 49] says that the set of extremal states is always a Baire space, i.e. the Baire Category Theorem holds for $\partial_e \mathcal{S}(M)$.

Let f be a real-valued function on $\mathcal{S}(M)$. We define

$$N(f) := \{s \in \partial_e \mathcal{S}(M) : f(s) \neq 0\}. \quad (3.1)$$

The proof of the Loomis-Sikorski Theorem from [BCD, Thm 4.1] used an effect-tribe \mathcal{T} of fuzzy sets defined on $\Omega := \mathcal{S}(M)$, and the effect-tribe \mathcal{T} is the class of all fuzzy sets f on $\mathcal{S}(M)$ with the property that there exists $b \in M$ such that $N(f - \hat{b})$ is a meager subset of $\partial_e \mathcal{S}(M)$ (in the relative topology), then we write $f \sim b$. The σ -homomorphism h was then defined by $h(f) := b$ if $f \sim b$. We call this triple (Ω, \mathcal{T}, h) the *canonical representation* of M . Every triple (Ω, \mathcal{T}, h) such that

\mathcal{T} is an effect-tribe of fuzzy sets on Ω and h maps σ -homomorphically \mathcal{T} onto M is said to be a *representation* of M .

Proposition 3.3. *Let (Ω, \mathcal{T}, h) be the canonical representation of a monotone σ -complete effect algebra M with RDP.*

- (i) *If $a \leq b$, $a, b \in M$, there are $f, g \in \mathcal{T}$ such that $f \leq g$ and $h(f) = a$, $h(g) = b$.*
- (ii) *If $f, g \in \mathcal{T}$, $f \leq g$, and let c be an element of M such that $h(f) \leq c \leq h(g)$. Then there exists a function $s \in \mathcal{T}$ such that $f \leq s \leq g$ and $h(s) = c$.*

Proof. (i) Let $f \sim a$ and $g \sim b$ for some $f, g \in \mathcal{T}$. We have $\max\{f, g\} \sim a \vee b = b$, which entails $\max\{f, g\} \in \mathcal{T}$ and $\max\{f, g\} \sim b$. In a similar way, $\min\{f, g\} \in \mathcal{T}$ and $\min\{f, g\} \sim a$.

(ii) Since h is surjective, there is a function $s_1 \in \mathcal{T}$ such that $h(s_1) = c$. If we set $s = \max\{f, \min\{g, s_1\}\}$, by (1), $s \in \mathcal{T}$ and s is the function in question. \square

Let \mathcal{T} be an effect-tribe on Ω and let $\mathcal{B}_0(M) := \{A \subseteq \Omega : \chi_A \in \text{Sh}(\mathcal{T})\}$. By [Dvu3, Prop 4.2], $\mathcal{B}_0(\mathcal{T})$ is a σ -algebra of subsets of Ω . Let $\mathcal{S}_0(\mathcal{T}) := \{A \subseteq \Omega : \chi_A \in \mathcal{T}\}$. If \mathcal{T} satisfies RDP, then $\mathcal{B}_0(\mathcal{T}) = \mathcal{S}_0(\mathcal{T})$, [Dvu3, p. 72]. By [Dvu3, Ex. 4.3], there is an effect-tribe \mathcal{T}' with RDP such that not every $f \in \mathcal{T}$ is $\mathcal{B}_0(\mathcal{T})$ -measurable. However, if \mathcal{T} is a tribe, then every $f \in \mathcal{T}$ is $\mathcal{B}_0(\mathcal{T})$ -measurable, [BuKl]. On the other hand, by [Dvu3, Prop 4.7] if an effect-tribe \mathcal{T} satisfies RDP, then \mathcal{T}' , the system of all functions $f \in \mathcal{T}$ such that f is $\mathcal{B}_0(\mathcal{T})$ -measurable, is an effect-tribe and $\mathcal{B}_0(\mathcal{T}) = \mathcal{B}_0(\mathcal{T}')$.

Theorem 3.4. *The canonical representation of a monotone σ -complete effect algebra M with RDP has the property $h(f) = 0$ if and only if $\chi_{N(f)} \in \mathcal{T}$ and $h(\chi_{N(f)}) = 0$. In addition, h maps $\mathcal{B}_0(\mathcal{T})$ onto $\text{Sh}(M)$.*

Proof. Let (Ω, \mathcal{T}, h) be the canonical representation of M used from [BCD, Thm 4.1]. We have that \mathcal{T} consists of all functions $f \in [0, 1]^\Omega$ such that $f \sim b$ for some $b \in M$; where we have $\Omega = \mathcal{S}(M)$ and $\Omega_0 = \partial_e \mathcal{S}(E)$. Assume that for $f \in \mathcal{T}$, we have $h(f) = 0$. This means that $f \sim 0$, that is, $N(f)$ is a meager set. Hence, $\{\omega \in \Omega_0 : \chi_{N(f)}(\omega) \neq 0\} = \{\omega \in \Omega_0 : f(\omega) \neq 0\}$ is meager. Whence, $\chi_{N(f)} \in \mathcal{T}$ and $\chi_{N(f)} \sim 0$ and $h(\chi_{N(f)}) = 0$.

Conversely, let for some $f \in \mathcal{T}$, $\chi_{N(f)} \in \mathcal{T}$, and $h(\chi_{N(f)}) = 0$. That is, $\chi_{N(f)} \sim 0$, and $N(f)$ is meager which entails $f \sim 0$.

Let $f = \chi_A \in \mathcal{B}_0(\mathcal{T})$ and let $g \in M$ be such that $g \leq h(f)$ and $g \leq h(f')$. Assume that $g_1 \in \mathcal{T}$ be such that $h(g_1) = g$. By (2) of Proposition 3.3, the functions $g_2 := \min\{f, g_1\}$ and $g_3 := \min\{f', g_1\}$ belong to \mathcal{T} , and $h(g_2) = g = h(g_3)$. Again by (2) of Proposition 3.3, the function $g_4 = \min\{g_2, g_3\} \in \mathcal{T}$ and $h(g_4) = g$. But $g_4 \leq f$ and $g_4 \leq 1 - f$ so that $g_4 = 0$ and $h(g_4) = g = 0$, and $h(f) \in \text{Sh}(M)$.

Now let $b \in \text{Sh}(M)$. Then $s(b) \in \{0, 1\}$ for any extremal state s on M . Define a function f_b on Ω by $f_b(s) = s(b)$ if s is an extremal state, otherwise, $f_b(s) := 0$. Then $f_b \sim b$, $f_b \in \mathcal{B}_0(\mathcal{T})$, and $h(f_b) = 0$. \square

The last result can be generalized as follows.

Let Ω_0 be a subset of a set $\Omega \neq \emptyset$. We recall that a σ -ideal on Ω_0 is a non-empty system \mathcal{I} of subsets of Ω_0 such that (i) if $A \subseteq B \in \mathcal{I}$, then $A \in \mathcal{I}$, and (ii) if $A_n \in \mathcal{I}$, $n \geq 1$, then $\bigcup_n A_n \in \mathcal{I}$. For example, let M be an effect algebra and let $\Omega = \mathcal{S}(M)$ and $\Omega_0 = \partial_e \mathcal{S}(M)$. The set of all meagre subsets of Ω_0 is a σ -ideal.

Let f be a real-valued function on Ω . We define

$$N_{\Omega_0}(f) := \{\omega \in \Omega_0 : f(\omega) \neq 0\}.$$

Let (Ω, \mathcal{T}, h) be a representation of a monotone σ -complete effect algebra M . We say that (Ω, \mathcal{T}, h) is *regular* if $h(f) = 0$ iff $\chi_{N_{\Omega_0}(f)} \in \mathcal{T}$ and $h(\chi_{N_{\Omega_0}(f)}) = 0$.

Theorem 3.5. *Let (Ω, \mathcal{T}, h) be a representation of a monotone σ -complete effect algebra M with RDP and let \mathcal{T} have RDP. Let \mathcal{I}_{Ω_0} be an ideal of subsets of a fixed subset Ω_0 of Ω such that $f \in [0, 1]^\Omega$ belongs to \mathcal{T} if and only if there exists a function $g \in \mathcal{T}$ such that $N_{\Omega_0}(f - g) \in \mathcal{I}_{\Omega_0}$. Then (Ω, \mathcal{T}, h) is regular and $h(f) = h(g)$ if and only if $N_{\Omega}(f - g) \in \mathcal{I}_{\Omega}$.*

In addition, (1) suppose every $f \in \mathcal{T}$ is $\mathcal{B}_0(\mathcal{T})$ -measurable, and if $h(f) \leq h(g)$, then $h(\chi_A) = 0$, where

$$A := \{\omega \in \Omega : f(\omega) > g(\omega)\}.$$

Then $h(\mathcal{B}_0(\mathcal{T})) \subseteq \text{Sh}(M)$.

(2) If $f \wedge (1 - f) \in \mathcal{T}$ for every $f \in \mathcal{T}$, then $h(\mathcal{B}_0(\mathcal{T})) = \text{Sh}(M)$.

Proof. Let $h(f) = 0$. Since $h(0) = 0$ and $0 \in \mathcal{T}$, we have $N_{\Omega_0}(f) = N_{\Omega_0}(f - 0) \in \mathcal{I}_{\Omega_0}$. Therefore, $N_{\Omega_0}(\chi_{N_{\Omega_0}(f)} - 0) = N_{\Omega_0}(\chi_{N_{\Omega_0}(f)}) = N_{\Omega_0}(f) \in \mathcal{I}_{\Omega_0}$, which entails $\chi_{N_{\Omega_0}(f)} \in \mathcal{T}$ and $h(\chi_{N_{\Omega_0}(f)}) = 0$.

Conversely, let for $f \in [0, 1]^\Omega$ we have $\chi_{N_{\Omega_0}(f)} \in \mathcal{T}$ and $h(\chi_{N_{\Omega_0}(f)}) = 0$. Then $N_{\Omega_0}(f) = N_{\Omega_0}(\chi_{N_{\Omega_0}(f)}) \in \mathcal{I}_{\Omega_0}$ which implies $f \in \mathcal{T}$. Set $f_0 = \max\{f, \chi_{N_{\Omega_0}(f)}\}$. Then $f_0 = \chi_{N_{\Omega_0}(f)}$ and $N_{\Omega_0}(f_0 - f) \subseteq N_{\Omega_0}(f)$ which yields $N_{\Omega_0}(f_0 - f) \in \mathcal{I}_{\Omega_0}$. Hence, $f_0 - f \in \mathcal{T}$ and $0 = h(f_0 - f) = h(f_0) - h(f) = 0 - h(f)$ from which we get $h(f) = 0$, and (Ω, \mathcal{T}, h) is a regular representation.

Now let $f, g \in \mathcal{T}$ are such that $N_{\Omega_0}(f - g) \in \mathcal{I}_{\Omega_0}$. We assert $h(f) = h(g)$. Indeed, define $g_0 = \max\{f, g\}$. Then $N_{\Omega_0}(g_0 - f) \subseteq N_{\Omega_0}(f - g)$ and whence, $N_{\Omega_0}(g_0 - f) \in \mathcal{I}_{\Omega_0}$, which yields $g_0 \in \mathcal{T}$, $h(g_0 - f) = 0$ and $h(g_0) = h(f)$. Similarly $N_{\Omega_0}(g_0 - g) \in \mathcal{I}_{\Omega_0}$ and $h(g_0) = h(g)$. Therefore, $h(f) = h(g_0) = h(g)$.

For the rest of the proof assume that every $f \in \mathcal{T}$ is $\mathcal{B}_0(\mathcal{T})$ -measurable and condition (1) of our hypotheses holds.

Claim 1. *If $a \leq b$, $a, b \in M$, there are $f, g \in \mathcal{T}$ such that $f \leq g$ and $h(f) = a$, $h(g) = b$.*

Let $h(f) = a$, $h(g) = b$. If $A = \{\omega \in \Omega : f(\omega) > g(\omega)\}$, by the assumption, $A \in \mathcal{B}_0(\mathcal{T})$ and $h(\chi_A) = 0$. By [Dvu3, Lem 4.1], for any $A \in \mathcal{B}_0(\mathcal{T})$ and any $f \in \mathcal{T}$, $f\chi_A = \min\{f, \chi_A\} \in \mathcal{T}$, where $f\chi_A$ means the product of two functions.

Define $f_0 = \max\{f, g\}$. If $\omega \in A$, $f_0(\omega) = f(\omega)$ and if $\omega \in A^c$, then $f_0(\omega) = g(\omega)$. Therefore, $N_{\Omega_0}(f_0\chi_A - f\chi_A) = \{\omega \in A \cap \Omega_0 : f(\omega) > f(\omega)\} = \emptyset \in \mathcal{I}_{\Omega}$ and $N_{\Omega_0}(f_0\chi_{A^c} - g\chi_{A^c}) = \{\omega \in A \cap \Omega_0 : g(\omega) > g(\omega)\} = \emptyset \in \mathcal{I}_{\Omega_0}$. Hence, $f\chi_A, f\chi_{A^c} \in \mathcal{T}$, $h(f_0\chi_A) = h(f\chi_A)$, $h(f_0\chi_{A^c}) = h(g\chi_{A^c})$ and consequently, $f_0 = f_0\chi_A + f_0\chi_{A^c} \in \mathcal{T}$, and $f_0 = f\chi_A + g\chi_{A^c}$.

Calculate: $h(f_0) = h(f\chi_A) + h(g\chi_{A^c})$. But $h(f\chi_A) = h(f \wedge \chi_A) \leq h(\chi_A) = 0$, and $h(g\chi_{A^c}) = h(g) - h(g\chi_A) = h(g)$ which gets $h(f_0) = h(g)$.

In the same way we can show that $g_0 = \min\{f, g\} \in \mathcal{T}$, $h(g_0\chi_A) = h(g\chi_A)$, $h(g_0\chi_{A^c}) = h(f\chi_{A^c})$, and $h(g_0) = h(f)$.

Claim 2. *If $f, g \in \mathcal{T}$, $f \leq g$, and let c be an element of M such that $h(f) \leq c \leq h(g)$. Then there exists a function $s \in \mathcal{T}$ such that $f \leq s \leq g$ and $h(s) = c$.*

Since h is onto, there is a function $s_1 \in \mathcal{T}$ such that $h(s_1) = c$. If we set $s = \max\{f, \min\{g, s_1\}\}$, by Claim 1, $s \in \mathcal{T}$ and s is the function in question.

Now we assume $f \in \mathcal{B}_0(\mathcal{T})$, and let $b \in M$ be such that $b \leq h(f), 1 - h(f)$.

Choose a function $g_1 \in \mathcal{T}$ such that $h(g_1) = b$. By Claim 2, the functions $g_2 := \min\{f, g_1\} \in \mathcal{T}$ and $g_3 := \min\{f', g_1\} \in \mathcal{T}$, and $h(g_2) = g = h(g_3)$. Again applying Claim 2, the function $g_4 = \min\{g_2, g_3\} \in \mathcal{T}$ and $h(g_4) = g$. But $g_4 \leq f$ and $g_4 \leq 1 - f$ so that $g_4 = 0$ and $h(g_4) = g = 0$, and $h(f) \in \text{Sh}(M)$.

Finally assume (2), and let $b \in \text{Sh}(M)$ and choose $g \in \mathcal{T}$ such that $h(g) = b$. Let $f \in \mathcal{T}$ be any function $f \leq g, 1 - g$, then $h(f) \leq b, b'$ giving $h(f) = 0$. Since h is regular, $\chi_{N_{\Omega_0}(f)} \in \mathcal{B}_0(\mathcal{T})$ and $h(\chi_{N_{\Omega_0}(f)}) = 0$. Since $g_0 = \min\{g, 1 - g\} \in \mathcal{T}$ and $g_0 \leq g, 1 - g$, we have $h(\chi_{N_{\Omega_0}(g_0)}) = 0$, and $N_{\Omega_0}(g_0) \in \mathcal{I}_{\Omega_0}$. Set $G = \{\omega \in \Omega : g(\omega) = 1\}$. Then $N_{\Omega_0}(g - \chi_G) = \{\omega \in \Omega_0 : g(\omega) \neq \chi_G(\omega)\} = \{\omega \in \Omega_0 : g \neq 0\} \cap \{\omega \in \Omega_0 : g(\omega) \neq 1\} = N_{\Omega_0}(g_0) \in \mathcal{I}_{\Omega_0}$. This proves that h maps $\mathcal{B}_0(\mathcal{T})$ onto $\text{Sh}(M)$. \square

We recall that in the latter theorem, the conditions are satisfied e.g. if M is a σ -complete MV-algebra and \mathcal{T} is a tribe.

4. SMEARING OF OBSERVABLES AND SPECTRAL MEASURES

This section is the main body of the paper. It presents results concerning smearing of observables by a sharp observable and a spectral measure of a given element.

The notion of an observable can be literally extended to any σ -homomorphism of effect algebras $\xi : \mathcal{S} \rightarrow M$, where \mathcal{S} is a σ -algebra of subsets of a set Ω . An observable ξ is *sharp* if $\xi(\mathcal{S}) \subseteq \text{Sh}(M)$.

We recall that a state s on a monotone σ -complete effect algebra M is σ -additive if $\{a_n\} \nearrow a$ implies $s(a) = \lim_n s(a_n)$. Let $\mathcal{S}_\sigma(M)$ denote the system of σ -additive states on M . We recall that there is even a Boolean σ -algebra which has lot of states but no σ -additive state, [Sik].

Theorem 4.1. *Let M be a monotone σ -complete effect algebra with RDP having at least one σ -additive state and let (Ω, \mathcal{T}, h) be the canonical representation of M such that every $f \in \mathcal{T}$ is $\mathcal{B}_0(\mathcal{T})$ -measurable. There is a sharp observable ξ from $\mathcal{B}_0(\mathcal{T})$ into M such that given an observable x on M , $m \in \mathcal{S}_\sigma(M)$ and $E \in \mathcal{B}(\mathbb{R})$*

$$m(x(E)) = \int_{\Omega} f_E(\omega) \, dm \circ \xi(\omega), \quad (4.1)$$

where f_E is an arbitrary function from \mathcal{T} such that $h(f_E) = x(E)$.

Proof. Let m be a σ -additive state on M , then $m \circ h$ is a σ -additive state on \mathcal{T} . By the generalized theorem of Klement and Butnariu holding for effect-tribes with condition that every $f \in \mathcal{T}$ is $\mathcal{B}_0(\mathcal{T})$ -measurable, [Dvu2, Thm 4.4], for every σ -additive state s on \mathcal{T} , there is a unique probability measure, P_s , on $\mathcal{B}_0(\mathcal{T})$ such that

$$s(f) = \int_{\Omega} f(\omega) \, dP_s(\omega), \quad f \in \mathcal{T}. \quad (4.2)$$

Let $x : \mathcal{B}_0(\mathbb{R}) \rightarrow M$ be an observable. Given $E \in \mathcal{B}_0(\mathbb{R})$, there is an element $f_E \in \mathcal{T}$ such that $h(f_E) = x(E)$. Using (4.3), given $m \in \mathcal{S}_\sigma(M)$, there is a unique

probability measure P_m on $\mathcal{B}_0(\mathcal{T})$ such that

$$m(x(E)) = m(h(f_E)) = \int_{\Omega} f_E(\omega) \, dP_m(\omega).$$

We assert the latter integral does not depend on the choice of f_E . Indeed, if g_E is another function from \mathcal{T} such that $h(g_E) = x(E)$, by (2) of Proposition 3.3, the function $h_E := \max\{f_E, g_E\}$ belongs to \mathcal{T} and $h(h_E) = x(E)$. But $h_E - f_E \in \mathcal{T}$ and $h(h_E - f_E) = 0$ so that

$$0 = m(h(h_E - f_E)) = \int_{\Omega} (h_E(\omega) - f_E(\omega)) \, dP_m(\omega).$$

In the similar way, we have

$$0 = m(h(h_E - g_E)) = \int_{\Omega} (h_E(\omega) - g_E(\omega)) \, dP_m(\omega),$$

whence

$$\int_{\Omega} f_E(\omega) \, dP_m(\omega) = \int_{\Omega} g_E(\omega) \, dP_m(\omega).$$

We assert that $P_m = m \circ h$. Indeed, let $f = \chi_A$, $A \in \mathcal{T}$. Then by (4.2),

$$m(h(\chi_A)) = \int_{\Omega} \chi_A(\omega) \, dP_m(\omega) = P_m(A).$$

The mapping $\xi : \mathcal{B}_0(\mathcal{T}) \rightarrow M$ defined by $\xi(A) := h(\chi_A)$, $A \in \mathcal{B}_0(\mathcal{T})$, is a sharp observable on M . \square

Commenting Theorem 4.1, we say that the observable x on M is a *smearing* of a sharp observable ξ . This result extends an analogous result for σ -lattice effect algebras, see [JPV1, Thm 3.4].

Remark 4.2. *Let (Ω, \mathcal{T}, h) be a regular representation of a monotone σ -complete effect algebra with RDP such that all conditions of Theorem 3.5 are satisfied. Then Theorem 4.1 holds also for our case of (Ω, \mathcal{T}, h) .*

Proof. It follows the same steps as the proof of Theorem 4.1. \square

Theorem 4.3. *Let M be a monotone σ -complete effect algebra with RDP and let (Ω, \mathcal{T}, h) be the canonical representation of M such that every $f \in \mathcal{T}$ is $\mathcal{B}_0(\mathcal{T})$ -measurable. Given $a \in M$, there is a mapping $\Lambda_a : \mathcal{B}_0([0, 1]) \rightarrow \text{Sh}(M)$ such that the mapping $a \mapsto \Lambda_a$ is injective, and for every σ -additive state m on M , we have*

$$m(a) = \int_0^1 \lambda \, dm(\Lambda_a(\lambda)). \quad (4.3)$$

Proof. Let (Ω, \mathcal{T}, h) be the canonical representation of M . Given $a \in M$, choose a function $f = f_a \in \mathcal{T}$ such that $h(f) = a$. For any Borel set $E \in \mathcal{B}_0([0, 1])$, let

$$\Lambda_a(E) := h(\chi_{f_a^{-1}(E)}). \quad (4.4)$$

Then Λ_a is by Theorem 3.4 an observable on $\text{Sh}(M)$.

Assume that $g \in \mathcal{T}$ is another function such that $h(g) = a$. As in the proof of Theorem 4.1, we can find a function $k \in \mathcal{T}$ such that $h(k) = a$ and $f, g \leq k$. Then $N(k - g)$ and $N(k - f)$ are meager sets, $k - f, k - g \in \mathcal{T}$, and $\chi_{f^{-1}(E)} - \chi_{k^{-1}(E)} \in \mathcal{T}$, $\chi_{g^{-1}(E)} - \chi_{k^{-1}(E)} \in \mathcal{T}$ for any $E = [0, t)$, $0 \leq t \leq 1$. In addition, $\chi_{f^{-1}(E)} - \chi_{k^{-1}(E)} \sim 0$, $\chi_{g^{-1}(E)} - \chi_{k^{-1}(E)} \sim 0$ for any $E = [0, t)$, $0 \leq t \leq 1$.

Therefore, $h(\chi_{f^{-1}(E)}) = h(\chi_{k^{-1}(E)})$ for every $E = [0, t]$. Let \mathcal{K} be the set of Borel subsets E from $[0, 1]$ such that $h(\chi_{f^{-1}(E)}) = h(\chi_{k^{-1}(E)})$. It is a Dynkin system, i.e. a system of subsets containing its universe which is closed under the set theoretical complements and countable unions of disjoint subsets.

The system \mathcal{K} contains all intervals $(-\infty, t) \cap [0, 1]$ for $t \in \mathbb{R}$, all intervals of the form $[a, b) \cap [0, 1]$, $a \leq b$, as well as all finite unions of such disjoint intervals $\bigcup_{i=1}^n [a_i, b_i) \cap [0, 1]$. Because any finite union of intervals $\bigcup_{j=1}^m [c_j, d_j) \cap [0, 1]$ can be expressed as a finite union of disjoint intervals, \mathcal{K} contains also such unions. Therefore, if E and F are two finite unions of intervals, so is its intersection. Hence, by [Dvu, Thm 2.1.10], \mathcal{K} is also a σ -algebra, and finally we have $\mathcal{K} = \mathcal{B}_0([0, 1])$. In the same way, $h(\chi_{g^{-1}(E)}) = h(\chi_{k^{-1}(E)})$ and, consequently, $h(\chi_{f^{-1}(E)}) = h(\chi_{g^{-1}(E)})$ for every $E \in \mathcal{B}_0([0, 1])$.

Consequently, we have proved that Λ_a in (4.4) does not depend on the choice of f .

Now assume that $\Lambda_a = \Lambda_b$ for some $a, b \in M$ and let $h(f) = a$ and $h(g) = b$ for some $f, g \in \mathcal{T}$. Then for every $E \in \mathcal{B}_0([0, 1])$, $h(\chi_{f^{-1}(E)}) = h(\chi_{g^{-1}(E)})$.

$$\begin{aligned} N(f - g) &= \{s \in \mathcal{S}_\partial(M) : f(s) \neq g(s)\} = \bigcup_{r \in \mathbb{Q}} \{s \in \mathcal{S}_\partial(M) : f(s) < r < g(s)\} \\ &= \bigcup_{r \in \mathbb{Q}} (f^{-1}([0, r)) \cap g^{-1}([r, 1])) = \bigcup_{r \in \mathbb{Q}} f^{-1}([0, r)) \Delta g^{-1}([0, r)), \end{aligned}$$

where Δ denotes the symmetric difference of two sets. Since $h(\chi_{f^{-1}([0, r))}) = h(\chi_{g^{-1}([0, r))})$ for every rational $r \in [0, 1]$, $N(\chi_{f^{-1}([0, r))} - \chi_{g^{-1}([0, r))})$ is a meager set. But $N(\chi_{f^{-1}([0, r))} - \chi_{g^{-1}([0, r))}) = f^{-1}([0, r)) \Delta g^{-1}([0, r))$, so that $N(f - g)$ is a meager set, and hence $a = h(f) = h(g) = b$.

Now let m be an arbitrary σ -additive state on M . It is clear that the mapping $m \circ h$ is a σ -additive state on \mathcal{T} , and $m_h : \mathcal{B}_0(\mathcal{T}) \rightarrow [0, 1]$, defined by $m_h(A) = m(h(\chi_A))$, $A \in \mathcal{B}_0(\mathcal{T})$, is a probability measure on $\mathcal{B}_0(\mathcal{T})$. Given an element $a \in M$, there is an element $f_a \in \mathcal{T}$ such that $h(f_a) = a$. Whence, the mapping $E \mapsto m(h(\chi_{f_a^{-1}(E)}))$, $E \in \mathcal{B}_0([0, 1])$, is a probability measure on $\mathcal{B}_0([0, 1])$. By [Dvu3, Thm 4.4], there is a unique probability measure P_m on $\mathcal{B}_0(\mathcal{T})$ such that

$$m(h(f)) = \int_{\Omega} f(\omega) \, dP_m(\omega), \quad f \in \mathcal{T}.$$

On the other hand, if $A \in \mathcal{B}_0(\mathcal{T})$, then

$$m(h(\chi_A)) = \int_{\Omega} \chi_A(\omega) \, dP_m(\omega) = \int_{\Omega} \chi_A(\omega) \, dm_h(\omega).$$

Since every $f \in \mathcal{T}$ is $\mathcal{B}_0(\mathcal{T})$ -measurable, we have by [Dvu3, Thm 4.4], that $P_m(A) = m_h(A)$ for every $A \in \mathcal{B}_0(\mathcal{T})$.

This yields for $f = f_a$

$$m(h(f_a)) = \int_{\Omega} f_a(\omega) \, dm_h(\omega) = \int_0^1 \lambda \, dm(h(\chi_{f_a^{-1}(\lambda)})) = \int_0^1 \lambda \, dm(\Lambda_a(\lambda)),$$

and

$$m(a) = \int_0^1 \lambda \, dm(\Lambda_a(\lambda))$$

which proves (4.3). \square

Theorem 4.3 generalizes an analogous result from [Pul], and the mapping $a \mapsto \Lambda_a$ is said to be the *spectral measure* of the element a .

Remark 4.4. *Let the conditions of Theorem 4.3 be satisfied.*

(1) *If $a \in \text{Sh}(M)$, then for Λ_a defined by (4.4), we have*

$$\Lambda_a(E) = \begin{cases} a & \text{if } 0 \notin E, 1 \in E, \\ a' & \text{if } 0 \in E, 1 \notin E, \\ 0 & \text{if } 0, 1 \notin E, \\ 1 & \text{if } 0, 1 \in E, \end{cases} \quad E \in \mathcal{B}_0([0, 1]).$$

(2) *Let $\phi : [0, 1] \rightarrow [0, 1]$ be a strictly increasing and surjective Borel-measurable function such that $\phi(0) = 0$ and $\phi(1) = 1$. If we define $\phi(\Lambda_a) : \mathcal{B}_0([0, 1]) \rightarrow \text{Sh}(M)$, $a \in M$, by $\phi(\Lambda_a)(E) := \Lambda_a(\phi^{-1}(E))$, $E \in \mathcal{B}_0([0, 1])$, then the mapping $a \mapsto \phi(\Lambda_a)$ is injective, but $\phi(\Lambda_a)$ is not necessarily a spectral measure because not always*

$$m(a) = \int_0^1 \phi(\lambda) dm(\Lambda_a(\lambda)).$$

Proof. (1) Let $a \in \text{Sh}(M)$. By Theorem 3.4, there is $A \in \mathcal{B}_0(\mathcal{T})$ such that $h(\chi_A) = a$. By (4.4), we have $\Lambda_a(E) = h(\chi_{\chi_A^{-1}(E)})$, for $E \in \mathcal{B}_0([0, 1])$. If $E = \{1\}$, then $\Lambda_a(\{1\}) = h(\chi_A) = a$. Similarly for other Borel sets E .

(2) Let $a \in M$ and let $h(f) = a$. Using (4.4), we have $\phi(\Lambda_a)(E) = \Lambda_a(\phi^{-1}(E)) = h(\chi_{f^{-1}(\phi^{-1}(E))}) = h(\chi_{(\phi \circ f)^{-1}(E)})$. Assume now $\phi(\Lambda_a) = \phi(\Lambda_b)$ for some b and let $h(g) = b$. Then $h(\chi_{(\phi \circ f)^{-1}(E)}) = h(\chi_{(\phi \circ g)^{-1}(E)})$, $E \in \mathcal{B}_0([0, 1])$. Similarly as in the proof of Theorem 4.3, we have that the set

$$\begin{aligned} N(f - g) &= \{s \in \mathcal{S}_\partial(M) : f(s) \neq g(s)\} \\ &= \bigcup_{r \in \mathbb{Q}} \{s \in \mathcal{S}_\partial(M) : f(s) < r < g(s)\} \\ &= \bigcup_{r \in \mathbb{Q}} (f^{-1}(\phi^{-1}([0, \phi(r)))) \cap g^{-1}(\phi^{-1}([\phi(r), 1]))) \\ &= \bigcup_{r \in \mathbb{Q}} f^{-1}(\phi^{-1}([0, \phi(r)))) \Delta g^{-1}(\phi^{-1}([0, \phi(r)))). \end{aligned}$$

Hence, we have $h(\chi_{f^{-1}(\phi^{-1}([0, \phi(r)))))} = h(\chi_{g^{-1}(\phi^{-1}([0, \phi(r)))))}$ and $N(\chi_{f^{-1}(\phi^{-1}([0, \phi(r)))))} - \chi_{g^{-1}(\phi^{-1}([0, \phi(r)))))}$ is a meager set. But

$$N(\chi_{f^{-1}(\phi^{-1}([0, \phi(r)))))} - \chi_{g^{-1}(\phi^{-1}([0, \phi(r)))))} = f^{-1}(\phi^{-1}([0, \phi(r)))) \Delta g^{-1}(\phi^{-1}([0, \phi(r)))),$$

so that $N(f - g)$ is a meager set, and hence $a = h(f) = h(g) = b$.

On the other hand, for any σ -additive state m on M , we have $\int_0^1 \lambda dm(\phi(\Lambda_a(\lambda))) = \int_0^1 \phi(\lambda) dm(\Lambda_a(\lambda))$ which is not necessarily equal $m(a)$. \square

We note that we do not know whether the spectral measure is unique.

Theorem 4.5. *Let M be a monotone σ -complete effect algebra with RDP and let (Ω, \mathcal{T}, h) be the canonical representation of M such that every $f \in \mathcal{T}$ is $\mathcal{B}_0(\mathcal{T})$ -measurable. Then every σ -additive state m on the Boolean σ -algebra $\text{Sh}(M)$ can be*

uniquely extended to a σ -additive state \hat{m} on M . In addition,

$$\hat{m}(a) = \int_0^1 \lambda \, dm(\Lambda_a(\lambda)), \quad a \in M. \quad (4.5)$$

Proof. Existence. Let m be a σ -additive state on $\text{Sh}(M)$. Then the mapping $m_h(A) := m(h(\chi_A))$, $A \in \mathcal{B}_0(\mathcal{T})$, is due to Theorem 3.4 a σ -additive measure on $\mathcal{B}_0(\mathcal{T})$. Then the mapping $s_m : \mathcal{T} \rightarrow [0, 1]$ defined by

$$s_m(f) := \int_{\Omega} f(\omega) \, dm_h(\omega), \quad f \in \mathcal{T},$$

is a σ -additive state on \mathcal{T} . Now we define a function $\hat{m} : M \rightarrow [0, 1]$ via $\hat{m}(a) = s_m(f)$ whenever $h(f) = a$. We claim that \hat{m} is defined correctly. Indeed, if $h(g) = a$, by (2) of Proposition 3.3, there is a function $k \in \mathcal{T}$ such that $h(k) = a$ and $f, g \leq k$. Therefore, $N(k - f)$ and $N(k - g)$ are meager sets, and $k - f, k - g \in \mathcal{T}$. Then

$$s_m(k - f) = \int_{\Omega} (k(\omega) - f(\omega)) \, dm_h(\omega) = 0 = \int_{\Omega} (k(\omega) - g(\omega)) \, dm_h(\omega),$$

which entails $\int_{\Omega} f(\omega) \, dm_h(\omega) = \int_{\Omega} g(\omega) \, dm_h(\omega)$.

Assume $a + b$ is defined in M , again by (2) of Proposition 3.3, we can assume that we have two functions $f, g \in \mathcal{T}$ such that $f \leq 1 - g$, and $h(f) = a$, $h(g) = b$. Then $h(f + g) = a + b$ and

$$\hat{m}(a + b) := \int_{\Omega} (f(\omega) + g(\omega)) \, dm_h(\omega) = \hat{m}(a) + \hat{m}(b).$$

It is clear that \hat{m} is a state on M . To show that \hat{m} is σ -additive, assume $\{a_n\} \nearrow a$ in M . Using by (2) of Proposition 3.3 and mathematical induction, we can assume that we have find a monotone sequence, $\{f_n\}$, of elements of \mathcal{T} such that $h(f_n) = a_n$ and $h(a) = h(\bigvee_n f_n) = \bigvee_n a_n = a$. Therefore,

$$\hat{m}(a) = \int_{\Omega} f(\omega) \, dm_h(\omega) = \lim_n \int_{\Omega} f_n(\omega) \, dm_h(\omega) = \lim_n \hat{m}(a_n),$$

which proves \hat{m} is a σ -additive state on M . Let now $a \in \text{Sh}(M)$. By Theorem 3.4, there is $A \in \mathcal{B}_0(\mathcal{T})$ such that $h(\chi_A) = a$. Then

$$\hat{m}(a) = \int_{\Omega} \chi_A \, dm_h(\omega) = m_h(A) = m(h(\chi_A)) = m(a),$$

which says that \hat{m} is an extension of m onto M .

Uniqueness. Let m_1 and m_2 be two σ -additive extensions of m onto the whole M . Define $s_i(f) := m_i(h(f))$, $f \in \mathcal{T}$, $i = 1, 2$. Then s_i is a σ -additive state on \mathcal{T} such that $s_1(\chi_A) = m(h(\chi_A)) = s_2(\chi_A)$. By [Dvu3, Thm 4.4], there are two probability measures P_1 and P_2 on $\mathcal{B}_0(\mathcal{T})$ such that

$$s_i(f) = \int_{\Omega} f(\omega) \, dP_i(\omega), \quad f \in \mathcal{T}.$$

Then $s_1(\chi_A) = P_1(A) = m(h(\chi_A)) = P_2(A) = s_2(\chi_A)$. Therefore, $s_1(f) = s_2(f)$ and if given $a \in M$, $h(f) = a$ for some $f \in \mathcal{T}$, $m_1(a) = m_2(a)$.

Finally, let $a \in M$ be given. Choose $f \in \mathcal{T}$ such that $h(f) = a$. Then using the Integral Transformation Theorem [Hal], we have

$$m(a) = \int_{\Omega} f(\omega) \, dm_h(\omega) = \int_0^1 \lambda \, dm_h(f^{-1}(\lambda)) = \int_0^1 \lambda \, dm(\Lambda_a(\lambda)),$$

when we have used (4.3). \square

We recall that Theorem 4.5 cannot be, in general, extended for any (finitely additive) state on $\text{Sh}(M)$ of a monotone σ -complete effect algebra M with RDP because this is possible iff M is an MV-algebra as it was proved in [Dvu3, Thm 5.1].

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